Γ-CONVERGENCE OF SOME SUPER QUADRATIC FUNCTIONALS WITH SINGULAR WEIGHTS

GIAMPIERO PALATUCCI AND YANNICK SIRE

Abstract. We study the Γ-convergence of the following functional ($p > 2$)

$$F_\varepsilon(u) := \varepsilon^{p-2}\int_\Omega |Du|^p d(x,\partial\Omega)^a dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_\Omega W(u) d(x,\partial\Omega)^{-\frac{a}{p-1}} dx + \frac{1}{\sqrt{\varepsilon}}\int_{\partial\Omega} V(Tu) d\mathcal{H}^2,$$

where Ω is an open bounded set of $\mathbb{R}^3$ and W and V are two non-negative continuous functions vanishing at α, β and α', β', respectively. In the previous functional, we fix $a = 2 - p$ and u is a scalar density function, Tu denotes its trace on $\partial\Omega$, $d(x, \partial\Omega)$ stands for the distance function to the boundary $\partial\Omega$. We show that the singular limit of the energies F_ε leads to a coupled problem of bulk and surface phase transitions.

Contents

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1. Introduction

This paper is devoted to the $\Gamma-$convergence of the following functional ($p > 2$)

$$F_\varepsilon(u) := \varepsilon^{p-2}\int_\Omega |Du|^p d(x,\partial\Omega)^a dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_\Omega W(u) d(x,\partial\Omega)^{-\frac{a}{p-1}} dx + \frac{1}{\sqrt{\varepsilon}}\int_{\partial\Omega} V(Tu) d\mathcal{H}^2,$$

where Ω is a bounded set in $\mathbb{R}^3$, V, W are two non-negative continuous functions vanishing at α, β and α', β' respectively and a is a fixed number, equal to $a = 2 - p$; Tu denotes the trace of u on $\partial\Omega$.

A lot of work has been devoted to the analysis of the asymptotic behavior of the functional (see for instance [11, 12])

$$E_\varepsilon(u) := \varepsilon \int_\Omega |Du|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(u) dx. \tag{1.1}$$

In particular, Modica proved that the previous functional E_ε $\Gamma-$converges in L^1 to

$$E(u) = \sigma \mathcal{H}^2(Su)$$

among all the admissible configurations $u \in BV(\Omega; \{\alpha, \beta\})$ with fixed volume. In the previous functional E, σ is a constant depending only on the potential W and $\mathcal{H}^2(Su)$ is the surface measure of the complement of Lebesgue points of u.

In [3], Alberti, Bouchitté and Seppecher considered the so-called two-phase model related to capillarity energy with line tension

$$E_\varepsilon(u) := \varepsilon \int_\Omega |Du|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(u) dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu) d\mathcal{H}^2. \tag{1.2}$$

The case $\lambda_\varepsilon = \lambda$ has been considered by Modica (with V being a positive continuous function), while Alberti, Bouchitté and Seppecher considered a logarithmic scaling, namely $\varepsilon \log \lambda_\varepsilon \to K > 0$ as ε goes to 0. Our approach here is to consider another penalization by perturbing with the term

$$\int_\Omega |Du|^p d(x,\partial\Omega)^a dx.$$

When $a = 0$, this case has been considered by one of the authors (see [16, 17]). We consider the case when we add a weight to the gradient term, namely $d(x,\partial\Omega)$. This weight is somehow related to some non local problems involving fractional powers of the laplacian.

In the paper [6], Caffarelli and Silvestre proved that one can realize any power of the fractional laplacian operator $(-\Delta)^s$ via an $s-$harmonic extension in the half-space. The fractional laplacian $(-\Delta)^s$ ($s \in (0,1)$) is a pseudo-differential operator of symbol $|\xi|^{2s}$. Caffarelli and Silvestre proved the following result: consider the boundary Dirichlet problem (with $y \in \mathbb{R}^n$ and $x > 0$)

$$\begin{cases} \operatorname{div}(x^a \nabla v) = 0 & \text{on } \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty) \\ v = f, & \text{on } \mathbb{R}^n \times \{0\}, \end{cases} \tag{1.3}$$

where f is a given smooth compactly supported function (for instance) and v is of finite energy (namely $\int_{\mathbb{R}^{n+1}_+} x^a |\nabla v|^2 \, dx, dy < \infty$). Then, up to a normalizing factor, the Dirichlet-to-Neumann operator $\Gamma_a : v|_{\partial\mathbb{R}^{n+1}_+} \mapsto -x^a v_x|_{\partial\mathbb{R}^{n+1}_+}$ is precisely $(-\Delta)^{\frac{1-a}{2}}$. As a consequence, one has the following corollary (see [6]): let u be a solution of

$$(-\Delta)^s u(y) = f(y), \quad y \in \mathbb{R}^n$$

and consider P_s the Poisson kernel associated to the operator $\mathrm{div}(x^{1-2s}\nabla)$. Therefore, the function $v = P_s \star_y u$ is a solution of the following problem

$$\begin{cases} \mathrm{div}\,(x^a \nabla v) = 0 & \text{on } \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty) \\ v = u, & \text{on } \mathbb{R}^n \times \{0\}, \\ -x^a v_x = f & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{1.4}$$

Note that the condition $\frac{1-a}{2} = s \in (0,1)$ reduces to $a \in (-1,1)$. The weight x^a is a particular weight since it belongs to Muckenhoupt A_2 classes (see [14]). Indeed, since $a \in (-1,1)$, the weight x^a (as its inverse) is locally integrable.

A quick look at the weight x^a shows that it is just the distance of a point $(x,y) \in \mathbb{R}^{n+1}_+$ to the boundary of the domain, namely $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n$. Therefore, a natural generalization in bounded domains consists in taking as the weight the distance to the boundary $d(x,\partial\Omega)^a$. In this case, there are no results available to describe what is precisely the boundary operator. However, one can expect that such a weight produces some new geometrical effects.

In the present work, we concentrate on a quasi-linear functional F_ε, i.e. $p > 2$. The case $p = 2$ has been considered in [9]. In this case, the main point consists in replacing the penalizing term of the functional by its Sobolev trace norm. To be able to do such a trick, which goes back to [2], one has to consider the optimal Sobolev embedding, i.e. to use the optimal constant in the Sobolev inequality. Using Caffarelli-Silvestre extension technique, Gonzalez computed explicitely the constant of this embedding.

The case $p > 2$ involves more technicalities due to the quasi-linear feature of the perturbation. In particular, we do not know how to replace the penalizing term by a Sobolev trace norm. Another difficulty comes from the scaling property. Indeed, the super-quadratic case enjoys a natural scaling which forces the parameter a in the functional to be exactly $a = 2 - p$. As a consequence, as soon as $p \geq 3$, the weight $d(x,\partial\Omega)^a$ is no longer locally integrable. In this case, Nekvinda (see [15]) proved that functions of the weighted Sobolev space $W^{1,p}(\Omega, d(x,\partial\Omega)^a)$ have no trace on $\partial\Omega$. Therefore, one has to use new techniques to deal with this case. As a consequence of this, we will be constrained to the range $p \in (2,3)$.

To simplify notations, we will denote $h(x) = d(x,\partial\Omega)$ and then consider the following functional

$$F_\varepsilon(u) := \varepsilon^{p-2} \int_\Omega |Du|^p h^{2-p} dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_\Omega W(u) h^{\frac{p-2}{p-1}} dx + \frac{1}{\sqrt{\varepsilon}} \int_{\partial\Omega} V(Tu) d\mathcal{H}^2. \tag{1.5}$$

At this point, some remarks on the scaling have to be noticed. Choosing $\varepsilon^{\frac{p-2}{p-1}}$ to denote the length of the bulk transition, by standard scaling analysis, the power ε^{p-2} follows naturally in the perturbation term. The election of the square root of $1/\varepsilon$ in the boundary term is justified by the scaling property of the functional F_ε (see Section 4).

2. Description of the results

We first fix notations, recalling also some standard mathematical results used throughout the paper. Then, we analyze the asymptotic behavior of the functional F_ε defined in (1.5) stating the related main convergence result.

2.1. **Notation.** In this work, we consider different domains A in dimensions $n = 1, 2, 3$; more precisely, A will always be a bounded open set of $\mathbb{R}^n$. We denote by ∂A the boundary of A relative to the ambient space; ∂A is always assumed to be Lipschitz regular. Unless otherwise stated, A is endowed with the corresponding n-dimensional Hausdorff measure, $\mathcal{H}^n$ (see [7], Chapter 2). We write $\displaystyle\int_A f dx$ instead of $\displaystyle\int_A f d\mathcal{H}^n$.

The *essential boundary* of A is the set of all points where A has neither 0 nor 1 density and where the density does not exist. Since the essential boundary agrees with the topological boundary when the latter is Lipschitz regular, we also denote the essential boundary by ∂A.

For every $u \in L^1_{\rm loc}(A)$, we denote by Du the derivative of u in the sense of distributions. As usual, for every $p \geq 1$, $W^{1,p}(A)$ is the Sobolev space of all $u \in L^p(A)$ such that $Du \in L^p(A)$. Given a weight $w : A \to [0, \infty)$, and $p \geq 1$, we consider the weighted Sobolev space $W^{1,p}(A, w)$ the space of all functions u with norm

$$\|u\|^p_{W^{1,p}(A,w)} := \int_A |u|^p w dx + \int_A |Du|^p w dx.$$

$BV(A)$ is the space of all $u \in L^1(A)$ with bounded variation; i.e., such that Du is a bounded Borel measure on A. We denote by Su the *jump set*; i.e., the complement of the set of Lebesgue points of u.

For every $s \in (0, 1)$ and every $p \geq 1$, $W^{s,p}(A)$ is the space of all $u \in L^p(A)$ such that the fractional semi-norm $\displaystyle\int_A\!\!\int_A \frac{|u(x) - u(x')|^p}{|x - x'|^{sp+n}} dx dx'$ is finite.

We denote by T the trace operator which maps $BV(A)$ onto $L^1(\partial A)$ and $W^{1,p}(A, w)$ onto $W^{2-3/p,p}(\partial A)$, for a suitable weight w (see [15, Theorem 2.8]). In particular for $p \in (2, 3)$, there exists a constant S_p such that

$$\|Tu\|_{W^{2-3/p,p}(\partial\Omega)} \leq S_p \|u\|_{W^{1,p}(\Omega, d^{2-p}(x,\partial\Omega))} \quad \text{(see [15, Theorem 2.11]).}$$

For details and results about the theory of BV functions and Sobolev spaces we refer to [7], [4] and [1].

2.2. **The Γ-convergence result.** Let Ω be a bounded open subset of $\mathbb{R}^3$ with smooth boundary; let W and V be non-negative continuous functions on $\mathbb{R}$ with growth at least linear at infinity and vanishing respectively only in the "double well" $\{\alpha, \beta\}$, with $\alpha < \beta$, and $\{\alpha', \beta'\}$, with $\alpha' < \beta'$. Assume that the potential V is convex near its wells.

Let $p \in (2,3)$ be a real number. For every $\varepsilon > 0$ we consider the functional F_ε defined in $W^{1,p}(\Omega, h^{2-p})$, given by

$$F_\varepsilon(u) := \varepsilon^{p-2} \int_\Omega |Du|^p h^{2-p} dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_\Omega W(u) h^{\frac{p-2}{p-1}} dx + \frac{1}{\sqrt{\varepsilon}} \int_{\partial\Omega} V(Tu) d\mathcal{H}^2. \tag{2.1}$$

We analyze the asymptotic behavior of the functional F_ε in terms of Γ-convergence. Let (u_ε) be an equi-bounded sequence for F_ε; i.e., there exists a constant C such that $F(u_\varepsilon) \le C$. We observe that the term $\frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_\Omega W(u_\varepsilon) h^{\frac{p-2}{p-1}} dx$ forces u_ε to take values close to α and β, while the term $\varepsilon^{p-2} \int_\Omega |Du_\varepsilon|^p h^{2-p} dx$ penalizes the oscillations of u_ε. We will see that when ε tends to 0, the sequence (u_ε) converges (up to a subsequence) to a function u, belonging to $BV(\Omega)$, which takes only the values α and β. Moreover each u_ε has a transition from the value α to the value β in a thin layer close to the surface Su, which separates the bulk phases $\{u = \alpha\}$ and $\{u = \beta\}$. Similarly, the boundary term of F_ε forces the traces Tu_ε to take values close to α' and β', and the oscillations of the traces Tu_ε are again penalized by the integral $\varepsilon^{p-2} \int_\Omega |Du_\varepsilon|^p h^{2-p} dx$. Then, we expect that the sequence (Tu_ε) converges to a function v in $BV(\partial\Omega)$ which takes only the values α' and β', and that a concentration of energy occurs along the line Sv, which separates the boundary phases $\{v = \alpha'\}$ and $\{v = \beta'\}$.

In view of possible "dissociation of the contact line and the dividing line" (see [3, Example 5.2]), we recall that Tu may differ from v. Since the total energy $F_\varepsilon(u_\varepsilon)$ is partly concentrated in a thin layer close to Su (where u_ε has a transition from α to β), partly in a thin layer close to the boundary (where u_ε has a transition from Tu to v), and partly in the vicinity of Sv (where Tu_ε has a transition from α' to β'), we expect that the limit energy is the sum of a surface energy concentrated on Su, a boundary energy on $\partial\Omega$ (with density depending on the gap between Tu and v), and a line energy concentrated along Sv.

The asymptotic behavior of the functional F_ε is described by a functional Φ which depends on the two functions u and v. Let $\mathcal{W}$ be an antiderivative of $W^{(p-1)/p}$. For every $(u, v) \in BV(\Omega; \{\alpha, \beta\}) \times BV(\partial\Omega; \{\alpha', \beta'\})$, we will prove that

$$\Phi(u, v) := \sigma_p \mathcal{H}^2(Su) + c_p \int_{\partial\Omega} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2 + \gamma_p \mathcal{H}^1(Sv), \tag{2.2}$$

where as usual the jump sets Su and Sv are the complement of the set of Lebesgue points of u and v, respectively; c_p and σ_p are the constants defined by

$$c_p := \frac{p}{(p-1)^{(p-1)/p}}, \quad \sigma_p := c_p|\mathcal{W}(\beta) - \mathcal{W}(\alpha)|; \tag{2.3}$$

The constant γ_p is given by the optimal profile problem

$$\gamma_p := \inf \Bigg\{ \int_{\mathbb{R}^2_+} |Du|^p x_2^{2-p} dx + \int_{\mathbb{R}} V(Tu) d\mathcal{H}^1 : u \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+) : \tag{2.4}$$

$$\lim_{t\to-\infty} Tu(t) = \alpha', \ \lim_{t\to+\infty} Tu(t) = \beta' \Bigg\}. \tag{2.5}$$

Note that in the definition (2.4) we utilize the variables $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^+$ to denote any point of $\mathbb{R}^2_+$, so that we have always $h^{2-p} = x_2^{2-p}$.

The main convergence result is precisely stated in the following theorem.

Theorem 2.1. *Assume $p \in (2,3)$. Let $F_\varepsilon : W^{1,p}(\Omega, h^{2-p}) \to \mathbb{R}$ and $\Phi : BV(\Omega; \{\alpha,\beta\}) \times BV(\partial\Omega; \{\alpha',\beta'\}) \to \mathbb{R}$ defined by (2.1) and (2.2).*

Then

(i) [Compactness] *If $(u_\varepsilon) \subset W^{1,p}(\Omega, h^{2-p})$ is a sequence such that $F_\varepsilon(u_\varepsilon)$ is bounded, then $(u_\varepsilon, Tu_\varepsilon)$ is pre-compact in $L^1(\Omega) \times L^1(\partial\Omega)$ and every cluster point belongs to $BV(\Omega; \{\alpha,\beta\}) \times BV(\partial\Omega; \{\alpha',\beta'\})$.*

(ii) [Lower Bound Inequality] *For every $(u,v) \in BV(\Omega; \{\alpha,\beta\}) \times BV(\partial\Omega; \{\alpha',\beta'\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(\Omega, h^{2-p})$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ and $Tu_\varepsilon \to v$ in $L^1(\partial\Omega)$,*

$$\liminf_{\varepsilon\to0} F_\varepsilon(u_\varepsilon) \geq \Phi(u,v).$$

(iii) [Upper Bound Inequality] *For every $(u,v) \in BV(\Omega; \{\alpha,\beta\}) \times BV(\partial\Omega; \{\alpha',\beta'\})$ there exists a sequence $(u_\varepsilon) \subset W^{1,p}(\Omega, h^{2-p})$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$, $Tu_\varepsilon \to v$ in $L^1(\partial\Omega)$ and*

$$\limsup_{\varepsilon\to0} F_\varepsilon(u_\varepsilon) \leq \Phi(u,v).$$

We can easily rewrite this theorem in term of Γ-convergence. To this aim, we extend each F_ε to $+\infty$ on $L^1(\Omega) \setminus W^{1,p}(\Omega, h^{2-p})$ and, from Theorem 2.1, we deduce that

Corollary 2.2. *F_ε Γ-converges on $L^1(\Omega)$ to F, given by*

$$F(u) := \begin{cases} \inf\{\Phi(u,v) : v \in BV(\partial\Omega; \{\alpha',\beta'\})\} & \text{if } u \in BV(\Omega; \{\alpha,\beta\}), \\ +\infty & \text{elsewhere in } L^1(\Omega). \end{cases}$$

3. Strategy of the proof and some convergence results

The proof of Theorem 2.1 requires several steps in which we have to analyze different effects. Then, we can deduce the terms of the limit energy Φ, localizing three effects: the bulk effect, the wall effect and the boundary effect.

3.1. **The bulk effect.** In the bulk term, the limit energy can be evaluated like in [9]. This requires to generalize the Modica-Mortola results on the functional (1.1) (see [13]) to a functional with super-quadratic growth in the perturbation term involving the singular weight h^{2-p}.

For every open set $A \subset \mathbb{R}^3$, $p \in (2,3)$ and every real function $u \in W^{1,p}(A, h^{2-p})$, we consider the functional

$$G_\varepsilon(u,A) := \varepsilon^{p-2} \int_A |Du|^p h^{2-p} dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_A W(u) h^{\frac{p-2}{p-1}} dx. \tag{3.1}$$

Since there is no interaction with the boundary of A and the weight is regular in the interior, the asymptotic behavior of the functional G_ε will be very similar to the one of (1.1).

Theorem 3.1. *For every domain $A \subset \Omega$ the following statements hold.*

(i) *If $(u_\varepsilon) \subset W^{1,p}(A, h^{2-p})$ is a sequence with uniformly bounded energies $G_\varepsilon(u_\varepsilon, A)$. Then (u_ε) is pre-compact in $L^1(A)$ and every cluster point belongs to $BV(A; \{\alpha, \beta\})$.*

(ii) *For every $u \in BV(A; \{\alpha,\beta\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(A, h^{2-p})$ such that $u_\varepsilon \to u$ in $L^1(A)$,*

$$\liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, A) \geq \sigma_p \mathcal{H}^2(Su \cap A),$$

(iii) *For every $u \in BV(A; \{\alpha,\beta\})$ there exists a sequence $(u_\varepsilon) \subset W^{1,p}(A)$ such that $u_\varepsilon \to u$ in $L^1(A)$ and*

$$\limsup_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, A) \leq \sigma_p \mathcal{H}^2(Su \cap A).$$

Proof. The proof is close to the one of Gonzalez in [9, Proposition 3.1] and Modica-Mortola's one. Here we provide a sketch and the needed modifications due to the different growth power in the singular perturbation.

Using the following Young's inequality, $X, Y \geq 0$,

$$XY \leq \frac{X^p}{p} + \frac{Y^q}{q}, \quad \big(q : 1/p + 1/q = 1\big), \tag{3.2}$$

with

$$X = |Du| h^{\frac{2-p}{p}} p^{\frac{1}{p}} \varepsilon^{-\frac{2-p}{p}} \quad \text{and} \quad Y = W(u)^{\frac{1}{q}} h^{-\frac{2-p}{(p-1)q}} q^{\frac{1}{q}} \varepsilon^{-\frac{p-2}{(p-1)q}},$$

we obtain

$$G_\varepsilon(u,A) \geq c_p \int_A W^{\frac{p-1}{p}} |Du| dx = c_p \int_A |D(\mathcal{W}(u)| dx, \tag{3.3}$$

where $\mathcal{W}$ is a primitive of $W^{(p-1)/p}$ and c_p is defined by (2.3). This gives the compactness result (i) and the lower bound inequality (ii), using standard arguments.

Consider a function u in $BV(A;\{\alpha,\beta\})$. To construct the recovery sequence u_ε of the upper bound inequality (iii), we need to take care of the weight h^{2-p}.

First, without loss of generality, we may assume that the singular set Su of u is a Lipschitz surface in A ([10, Theorem 1.24]). For every x in A, let us define the signed distance from Su as

$$d'(x) := \begin{cases} \mathrm{dist}(x, Su) & \text{if } x \in \{u=\beta\}, \\ -\mathrm{dist}(x, Su) & \text{if } x \in \{u=\alpha\}. \end{cases}$$

We may consider smooth coordinates $(d'(x), \eta)$ in A such that η parametrizes Su.

Now, we choose $\theta \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ satisfying

$$\begin{cases} \theta' = \frac{1}{(p(p-1))^{1/p}} W^{1/p}(\theta) \;\; \text{a.e.} \\ \\ \theta(-\infty) = \alpha, \;\; \theta(+\infty) = \beta, \end{cases} \tag{3.4}$$

where the valus $\theta(\pm\infty)$ are understood as the existence of the corresponding limits. We remark that this real function θ is just the optimum profile for the case $a = 0$.

Consider the function $\phi : A \to \mathbb{R}$ defined by

$$\phi(t,\eta) \equiv \phi_\eta(t) := \theta\left(\frac{t}{h^{\frac{2-p}{p-1}}(0,\eta)}\right). \tag{3.5}$$

Finally, we are in position to construct the recovery sequence (u_ε). For every $\varepsilon > 0$, let $t = d'(x)/\varepsilon^{\frac{p-2}{p-1}}$ and

$$u_\varepsilon(x) := \phi_\eta\left(\frac{d'(x)}{\varepsilon^{\frac{p-2}{p-1}}}\right) \quad \forall x \in A.$$

Using the fact that for every $\delta \in (0,1)$ there exists $c(\delta) \to \infty$ when $\delta \to 0$ such that

$$(X+Y)^p \le (1+\delta)X^p + c(\delta)Y^p,$$

by definition of u_ε we have

$$\begin{aligned} |Du_\varepsilon|^p(x) &= \left|\frac{\partial \phi}{\partial t}(t(x),\eta)Dt(x) + \frac{\partial \phi}{\partial \eta}(t(x),\eta)\right|^p \\ &\le (1+\delta)\frac{(\phi'_\eta(t))^p}{\varepsilon^{\frac{p-2}{p-1}p}} + c(\delta)\mathcal{R}(\eta,t) \quad \forall x \in A, \end{aligned} \tag{3.6}$$

where we denoted by $\mathcal{R}(\eta,t) := \left|\dfrac{\partial\phi}{\partial\eta}(t,\eta)\right|^p$.

Thus we can estimate the energy of the function u_ε, using the CoArea Formula. For every $\delta \in (0,1)$ we have

$$\begin{aligned} G_\varepsilon(u_\varepsilon, A) &= \varepsilon^{p-2} \int_A |Du|^p h^{2-p} dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_A W(u) h^{\frac{p-2}{p-1}} dx \\ &\leq (1+\delta) \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_A \left[(\phi'_\eta(t))^p h^{2-p} + W(\phi_\eta(t)) h^{\frac{p-2}{p-1}} + c(\delta)\mathcal{R}(\eta,t)\varepsilon^{\frac{(p-2)p}{p-1}} \right] dx \\ &= (1+\delta) \int_{-\infty}^{+\infty} \int_{\Sigma_\varepsilon} \left[(\phi'_\eta(t))^p h^{2-p} + W(\phi_\eta(t)) h^{\frac{p-2}{p-1}} + c(\delta)\mathcal{R}(\eta,t)\varepsilon^{\frac{(p-2)p}{p-1}} \right] d\eta dt, \end{aligned}$$

with the level set $\Sigma_\varepsilon := \left\{ x \in A : d(x, Su) = \varepsilon^{\frac{p-2}{p-1}} t \right\}$ that converges to $Su \cap A$ when $\varepsilon \to 0$. Moreover, when ε goes to 0, $c(\delta)\mathcal{R}(\eta,t)\varepsilon^{\frac{(p-2)p}{p-1}}$ converges to 0 and, if x is written in the coordinates $(d'(x), \eta)$, then $h(t,\eta)$ converges to $\mathrm{dist}((0,\eta), \partial A) \equiv h(0,\eta)$. Hence, for every $\delta \in (0,1)$, taking the limit as ε goes to 0, we have

(3.7)
$$\limsup_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, A) \leq (1+\delta) \int_{-\infty}^{\infty} \int_{Su \cap A} \left[(\phi'_\eta(t))^p h^{2-p}(0,\eta) + W(\phi_\eta(t)) h^{\frac{p-2}{p-1}}(0,\eta) \right] d\eta dt.$$

Using the definitions of θ and ϕ_η given by (3.4) and (3.5), it follows that

$$\phi'_\eta(t) h^{\frac{2-p}{p}}(0,\eta) p^{\frac{1}{p}} = \left(W(\phi_\eta(t))^{\frac{1}{q}} h^{-\frac{2-p}{(p-1)q}} q^{\frac{1}{q}} \right)^{\frac{1}{p-1}}.$$

So, when we apply the inequality (3.2), like in (3.3), we also have an equality and the (3.7) becomes

$$\begin{aligned} \limsup_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, A) &\leq (1+\delta) \int_{Su \cap A} \int_{-\infty}^{+\infty} c_p W^{\frac{p-1}{p}}(\phi_\eta(t)) \phi'_\eta(t) dt d\eta \\ &= (1+\delta) \int_{Su \cap A} \int_{\alpha}^{\beta} c_p W^{\frac{p-1}{p}}(r) dr d\eta \\ &= (1+\delta)\sigma_p \mathcal{H}^2(Su \cap A) \quad \forall \delta \in (0,1). \end{aligned}$$

This concludes the proof. □

3.2. **The wall effect.** The second term of Φ can be obtained thanks to the following lemma.

Proposition 3.2. *For every domain $A \subset \Omega$ with boundary piecewise of class C^1 and for every $A' \subset \partial A$ with Lipschitz boundary, the following statements hold.*

(i) *For every $(u,v) \in BV(A; \{\alpha, \beta\}) \times BV(A'; \{\alpha', \beta'\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(A, h^{2-p})$ such that $u_\varepsilon \to u$ in $L^1(A)$ and $Tu_\varepsilon \to v$ in $L^1(A')$,*

$$\liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, A) \geq c_p \int_{A'} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2.$$

(ii) *Let a function v, constant on A', and a function u, constant on A, such that $u \equiv \alpha$ or $u \equiv \beta$, be given. Then there exists a sequence (u_ε) such that $Tu_\varepsilon = v$ on A', u_ε converges uniformly to u on every set with positive distance from A' and*

$$\limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, A) \le c_p \int_{A'} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2.$$

Moreover, the function u_ε may be required to be C_r-Lischitz continuous in $A_r := \{x \in A : d(x, \partial A) \le r\}$.

Proof. The proof of (i) is essentially contained in [12, Proposition 1.2 and Proposition 1.4], where Modica study a Cahn-Hilliard functional with quadratic growth in the singular perturbation term and with a boundary contribution (see also [16, Proposition 4.3] for details of the super-quadratic version). While, the proof of (ii) is very similar to [3, Proposition 4.3] and it can be recovered using the modifications introduced in the proof of Theorem 3.1-(iii). See also [9, Proposition 3.1] for the computation of the Lipschitz constant of u_ε.

3.3. **The boundary effect.** This is a delicate step, that requires a deeper analysis. The main strategy is the one used by Alberti, Bouchitté and Seppecher in [3] with the needed modifications introduced by one of the author in [17] for functionals with super-quadratic growth in the singular perturbation term. We reduce to the case in which the boundary is flat; hence we study the behavior of the energy in the three-dimensional half ball; then we reduce the problem to one dimension via a slicing argument.

Thus, the main problem becomes the analysis of the asymptotic behavior of the following two-dimensional functional

$$H_\varepsilon(u) := \varepsilon^{p-2} \int_{D_1} |Du|^p x_2^{2-p} dx + \frac{1}{\sqrt{\varepsilon}} \int_{E_1} V(Tu) d\mathcal{H}^1, \quad \forall u \in W^{1,p}(D_1, h^{2-p}), \tag{3.8}$$

where D_1 and E_1 are defined by

$$\begin{aligned} D_r &:= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r^2, x_2 > 0\right\}, \\ E_r &:= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r, x_2 = 0\right\} \equiv (-r, r). \end{aligned} \tag{3.9}$$

Note that for the quadratic case the two-dimensional Dirichlet weighted energy can be replaced on the half-disk D_r by the $H^{1/2}$ intrinsic norm on the "diameter" E_r. This is possible thanks to the existence of an optimal constant for the trace inequality involving the weighted L^2-norm of the gradient of a function defined on a two-dimensional domain and the $H^{1/2}$-norm of its trace on a line (see [9, Proposition 4]). Hence, the analysis of the line tension effect is reduced to the one of the following one-dimensional perturbation problem

involving a non-local term:

$$E_\varepsilon(v) = \varepsilon^{1-a} \iint_{I\times I} \frac{|v(t)-v(t')|^2}{|t-t'|^{1+2s}} dt'dt + \frac{1}{\varepsilon^{\frac{1-a}{-a}}} \int_I V(v)dt, \quad (I \text{ open interval of } \mathbb{R};\ s = (1-a)/2),$$

that was essentially studied by Garroni and one of the author in [8].

On the contrary, we have to study the asymptotic analysis of H_ε, that will be the subject of Section 4.

We conclude this section stating some properties of the functional F_ε.

3.4. **Some remark about the structure of F_ε.** The methods used in the proof of the main results of this paper strongly requires the "localization" of the functional F_ε; i.e., looking at F_ε as a function of sets. By fixing u we will be able to characterize the various effects of the problem. In this sense, for every open set $A \subset \mathbb{R}^3$, every set $A' \subseteq \partial A$ and every function $u \in W^{1,p}(A, h^{2-p})$, we will denote

$$F_\varepsilon(u, A, A') := \varepsilon^{p-2} \int_A |Du|^p h^{2-p} dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_A W(u) h^{\frac{p-2}{p-1}} dx + \frac{1}{\sqrt{\varepsilon}} \int_{A'} V(Tu) d\mathcal{H}^2. \tag{3.10}$$

Clearly, $F_\varepsilon(u) = F_\varepsilon(u, \Omega, \partial\Omega)$ for every $u \in W^{1,p}(\Omega, h^{2-p})$.

Let us observe that, thanks to the growth hypothesis on the potentials W and V, we may assume that there exists a constant m such that:

$$\begin{gathered} -m \le \alpha, \alpha', \beta, \beta' \le m, \\ W(t) \ge W(m) \text{ and } V(t) \ge V(m) \text{ for } t \ge m, \\ W(t) \ge W(-m) \text{ and } V(t) \ge V(-m) \text{ for } t \le -m. \end{gathered} \tag{3.11}$$

In particular, assumption (3.11) will allow us to use the truncation argument given by the following Lemma.

Lemma 3.3. *Let a domain $A \subset \mathbb{R}^3$, a set $A' \subseteq \partial A$, and a sequence $(u_\varepsilon) \subset W^{1,p}(A, h^{2-p})$ with uniformly bounded energies $F_\varepsilon(u_\varepsilon, A, A')$ be given.*
If we set $\bar{u}_\varepsilon(x) := \max\{\min\{u_\varepsilon(x), m\}, -m\}$, then

(i) $F_\varepsilon(\bar{u}_\varepsilon, A, A') \le F_\varepsilon(u_\varepsilon, A, A')$,
(ii) $\|\bar{u}_\varepsilon - u_\varepsilon\|_{L^1(A)}$ *and* $\|T\bar{u}_\varepsilon - Tu_\varepsilon\|_{L^1(A')}$ *vanish as* $\varepsilon \to 0$.

Proof. The inequality $F_\varepsilon(\bar{u}_\varepsilon, A, A') \le F_\varepsilon(u_\varepsilon, A, A')$ follows immediately from (3.11). Statement (ii) follows from the fact that both W and V have growth at least linear at infinity and the integrals $\int W(u_\varepsilon) h^{\frac{p-2}{p-1}} dx$ and $\int V(Tu_\varepsilon) d\mathcal{H}^2$ vanish as ε goes to 0. This is a standard argument; see, for instance, [16, Lemma 4.4]. □

4. Recovering the "contribution of the wall": the flat case

We will obtain "the contribution of the wall" to the limit energy Φ, defined by (2.2), namely $\gamma_p\mathcal{H}^1(Sv)$, by estimating the asymptotic behavior of the functional

$$F_\varepsilon(u,B\cap\Omega,B\cap\partial\Omega)=\varepsilon^{p-2}\int_{B\cap\Omega}|Du|^p h^{2-p}dx+\frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_{B\cap\Omega}W(u)h^{\frac{p-2}{p-1}}dx+\frac{1}{\sqrt{\varepsilon}}\int_{B\cap\partial\Omega}V(Tu)d\mathcal{H}^2,$$

when B is a small ball centered on $\partial\Omega$ and $B\cap\partial\Omega$ is a flat disk. We will follow the idea of Alberti, Bouchitté and Seppecher in [3], using a suitable slicing argument; the flatness assumption on $B\cap\partial\Omega$ can be dropped when B is sufficiently small. Hence, we need to prove a compactness result and a lower bound inequality for the following two-dimensional functional

$$H_\varepsilon(u):=\varepsilon^{p-2}\int_{D_1}|Du|^p h^{2-p}dx+\frac{1}{\sqrt{\varepsilon}}\int_{E_1}V(Tu)d\mathcal{H}^1,\quad \forall u\in W^{1,p}(D_1,h^{2-p};[-m,m]),\tag{4.1}$$

where E_1 and D_1 are defined by (3.9). We recall that we will always study H_ε like a reduction of F_ε. Hence there will be some hypotheses inherited by this reduction. In particular, the hypothesis $u\in[-m,m]$ in (4.1) is justified by Lemma 3.3.

Let us introduce the "localization" of the functional H_ε. For every open set $A\subset\mathbb{R}^2$, every set $A'\subset\partial A$ and every function $u\in W^{1,p}(A,h^{2-p})$, we will denote

$$H_\varepsilon(u,A,A'):=\varepsilon^{p-2}\int_A|Du|^p h^{2-p}dx+\frac{1}{\sqrt{\varepsilon}}\int_{A'}V(Tu)d\mathcal{H}^1.\tag{4.2}$$

Let $A=D_1$ be the half disk defined in (3.9) and denote by

$$D_1^0:=\{x=(x_1,x_2)\in D_1:\mathrm{dist}(x,\partial D_1)=\mathrm{dist}(x,E_1)\}.\tag{4.3}$$

If we set $u^{(\varepsilon)}(x):=u(x/\sqrt{\varepsilon})$ and $A/\sqrt{\varepsilon}:=\{x:\sqrt{\varepsilon}x\in A\}$, by scaling it is immediately seen that

$$\begin{aligned}\varepsilon^{p-2}\int_A|Du^{(\varepsilon)}|^p h^{2-p}dx &= \int_{D_1^0/\sqrt{\varepsilon}}|Du|^p x_2^{2-p}+\int_{(D_1^0)^c/\sqrt{\varepsilon}}|Du|^p\left(\frac{1}{\sqrt{\varepsilon}}-\sqrt{x_1^2+x_2^2}\right)^{2-p}dx\\ &=: I_1^\varepsilon+I_2^\varepsilon.\end{aligned}\tag{4.4}$$

Notice that at least formally I_1^ε tends to $\int_{\mathbb{R}^2_+}|Du|^p y_2^{2-p}dy$ as $\varepsilon\to0$ and we will control I_2^ε, under suitable assumptions. This is the object of Section 5.

In view of this scaling property, we consider the optimal profile problem, introduced in the Section 2.2; that is,

$$\begin{aligned}\gamma_p=\inf\Bigg\{\int_{\mathbb{R}^2_+}|Du|^p x_2^{2-p}dx+\int_{\mathbb{R}}V(Tu)d\mathcal{H}^1 : u\in L^1_{\mathrm{loc}}(\mathbb{R}^2_+):\\ \lim_{t\to-\infty}Tu(t)=\alpha',\ \lim_{t\to+\infty}Tu(t)=\beta'\Bigg\}\end{aligned}\tag{4.5}$$

and determines the line tension on the limit energy Φ.

4.1. **Compactness of the traces.** We prove the pre-compactness of the traces of the equi-bounded sequences for H_ε, using the trace embedding of $W^{1,p}(D_1, h^{2-p})$ in $W^{2-3/p,p}(\partial D_1)$ and the following lemma, which is an adaptation of [17, Lemma 4.1], using the estimations in [8, Lemma 4.1].

Lemma 4.1. *Let (u_ε) be a sequence in $W^{1,p}(D_1, h^{2-p}; [-m, m])$ and let $J \subset E_1$ be an open interval. For every δ such that $0 < \delta < (\beta' - \alpha')/2$, define*

$$A_\varepsilon := \{x \in E_1 : Tu_\varepsilon(x) \leq \alpha + \delta\} \text{ and } B_\varepsilon := \{x \in E_1 : Tu_\varepsilon(x) \geq \beta' - \delta\}$$

and set

$$a_\varepsilon := \frac{|A_\varepsilon \cap J|}{|J|} \text{ and } b_\varepsilon := \frac{|B_\varepsilon \cap J|}{|J|}. \tag{4.6}$$

Then

$$H_\varepsilon(u_\varepsilon, D_1, J) \geq \left(\frac{S_p(\beta - \alpha - 2\delta)^p}{(2p-3)(p-2)|J|^{2(p-2)}} \left(1 - \frac{1}{(1-a_\varepsilon)^{2(p-2)}} - \frac{1}{(1-b_\varepsilon)^{2(p-2)}}\right) - C_1 \right) \varepsilon^{p-2} + C_\delta, \tag{4.7}$$

where S_p, C_1 and C_δ are positive constants not depending on ε.

Proof. By the weighted Sobolev embedding of $W^{1,p}(D_1, h^{2-p})$ in $W^{2-3/p,p}(\partial D_1)$ (see [15, Theorem 2.11]), we have that there exists a constant S_p such that for every $u \in W^{1,p}(D_1, h^{2-p})$

$$\|Tu\|_{W^{2-3/p,p}(\partial D_1)} \leq S_p \|u\|_{W^{1,p}(D_1, h^{2-p})}.$$

It follows that there exists a constant (still denoted by S_p) such that

$$\int_{D_1} |Du|^p h^{2-p} dx \geq S_p \iint_{J \times J} \frac{|Tu(t) - Tu(t')|^p}{|t - t'|^{2(p-1)}} dt' dt - \int_{D_1} |u|^p h^{2-p} dx, \quad \forall u \in W^{1,p}(D_1, h^{2-p}).$$

Hence

$$
\begin{aligned}
H_\varepsilon(u_\varepsilon, D_1, J) &= \varepsilon^{p-2}\int_{D_1} |Du_\varepsilon|^p h^{2-p}dx + \frac{1}{\sqrt{\varepsilon}}\int_J V(Tu_\varepsilon)d\mathcal{H}^1 \\
&\geq S_p\varepsilon^{p-2}\iint_{J\times J}\frac{|Tu_\varepsilon(t)-Tu_\varepsilon(t')|^p}{|t-t'|^{2(p-1)}}dt'dt + \frac{1}{\sqrt{\varepsilon}}\int_J V(Tu_\varepsilon)d\mathcal{H}^1 \\
&\quad -\varepsilon^{p-2}\int_{D_1}|u_\varepsilon|^p h^{2-p}dx \\
&\geq S_p\varepsilon^{p-2}\iint_{J\times J}\frac{|Tu_\varepsilon(t)-Tu_\varepsilon(t')|^p}{|t-t'|^{2(p-1)}}dt'dt + \frac{1}{\sqrt{\varepsilon}}\int_J V(Tu_\varepsilon)d\mathcal{H}^1 - C_1\varepsilon^{p-2}.
\end{aligned} \tag{4.8}
$$

The remaining part of the proof follows as in [17, Lemma 4.1]. □

We are now in position to prove the compactness result stated in the following proposition.

Proposition 4.2. *If $(u_\varepsilon) \subset W^{1,p}(D_1, h^{2-p}; [-m,m])$ is a sequence such that $H_\varepsilon(u_\varepsilon)$ is bounded then (Tu_ε) is pre-compact in $L^1(E_1)$ and every cluster point belongs to $BV(E_1, \{\alpha', \beta'\})$.*

Proof. By hypothesis, there exists a constant C such that $H_\varepsilon(u_\varepsilon) \leq C$. In particular

$$\int_{E_1} V(Tu_\varepsilon)d\mathcal{H}^1 \leq C\sqrt{\varepsilon}$$

and this implies that

$$V(Tu_\varepsilon) \to 0 \text{ in } L^1(E_1). \tag{4.9}$$

Thanks to the growth assumptions on V, (Tu_ε) is equi-integrable. Hence, by Dunford-Pettis' Theorem, (Tu_ε) is weakly relatively compact in $L^1(E_1)$; i.e., there exists $v \in L^1(E_1)$ such that (up to subsequences) $Tu_\varepsilon \rightharpoonup v$ in $L^1(E_1)$.

We have to prove that this convergence is strong in $L^1(E_1)$ and that $v \in BV(E_1; \{\alpha', \beta'\})$. This proof is standard, involving Young measures associated to sequences (see also [2, Théorème 1-(i)]). Let ν_x be the Young measure associated with (Tu_ε). Since V is a non negative continuous function in $\mathbb{R}$, we have

$$\int_{E_1}\int_{\mathbb{R}} V(t)d\nu_x(t) \leq \liminf_{\varepsilon\to 0}\int_{E_1} V(Tu_\varepsilon)dx$$

(see [18, Theorem I.16]).

Hence, by (4.9), it follows that

$$\int_{\mathbb{R}} V(t)d\nu_x(t) = 0, \quad \text{a.e. } x \in E_1,$$

which implies the existence of a function θ on $[0,1]$ such that

$$\nu_x(dt) = \theta(x)\delta_{\alpha'}(dt) + (1-\theta(x))\delta_{\beta'}(dt), \quad x \in E_1$$

and
$$v(x) = \theta(x)\alpha' + (1 - \theta(x))\beta', \quad x \in E_1.$$
It remains to prove that θ belongs to $BV(E_1; \{0, 1\})$. Let us consider the set S of the points where approximate limits of θ is neither 0 nor 1. For every $N \leq \mathcal{H}^0(S)$ we can find N disjoint intervals $\{J_n\}_{n=1,\dots,N}$ such that $J_n \cap S \neq \emptyset$ and such that the quantities a^n_ε and b^n_ε, defined by (4.6) replacing J by J_n, satisfy
$$a^n_\varepsilon \to a^n \in (0, 1) \quad \text{and} \quad b^n_\varepsilon \to b^n \in (0, 1) \quad \text{as } \varepsilon \text{ goes to zero.}$$
We can now apply Lemma 4.1 in the interval J_n and, taking the limit as $\varepsilon \to 0$ in the inequality (4.7), we obtain
$$\liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon, D_1, J_n) \geq C_\delta.$$
Finally, we use the sub-additivity of the non local part of the functional and we get
$$\liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon, D_1, E_1) \geq \sum_{n=1}^{N} \liminf H_\varepsilon(u_\varepsilon, D_1, J_n) \geq NC_\delta.$$
Since (u_ε) has equi-bounded energy, this implies that S is a finite set. Hence, $\theta \in BV(E_1; \{0, 1\})$ and the proof of the compactness for H_ε is complete. □

4.2. **Lower bound inequality.** We will prove an optimal lower bound for H_ε.

Proposition 4.3. *For every (u, v) in $BV(D_1; \{\alpha, \beta\}) \times BV(E_1; \{\alpha', \beta'\})$ and every sequence $(u_\varepsilon) \subset W^{1,p}(D_1, h^{2-p}; [-m, m])$ such that $u_\varepsilon \to u$ in $L^1(D_1)$ and $Tu_\varepsilon \to v$ in $L^1(E_1)$*
$$\liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon) \geq \gamma_p \mathcal{H}^0(Sv). \tag{4.10}$$

Proof. We will prove the lower bound inequality (4.10) for v such that
$$v(t) = \begin{cases} \alpha', & \text{if } t \in (-1, 0], \\ \beta', & \text{if } t \in (0, 1). \end{cases}$$
Consider the natural extension of v to the whole real line $\mathbb{R}$, still denoted by v; that is
$$v(t) = \begin{cases} \alpha', & \text{if } t \leq 0, \\ \beta', & \text{if } t > 0. \end{cases}$$

Step 0: Strategy of the proof. We are looking for an extension of u_ε to the whole half-plane $\mathbb{R}^2_+$, namely w_ε, such that w_ε is a competitor for γ_p and $H_\varepsilon(w_\varepsilon, \mathbb{R}^2_+, \mathbb{R}) \simeq H_\varepsilon(u_\varepsilon, D_1, E_1)$ as $\varepsilon \to 0$ in a precise sense. For every $\varepsilon > 0$, we will be able to find $s < 1$ such that, for any given $\delta > 0$ there exists $\varepsilon_\delta > 0$ and we have
$$\begin{aligned} H_\varepsilon(u_\varepsilon) &\geq H_\varepsilon(u_\varepsilon, D_s, E_s) \\ &\geq \gamma_p - \delta, \quad \forall \varepsilon \leq \varepsilon_\delta. \end{aligned}$$

Step 1: Construction of the competitor. For every $s > 0$, we denote by $\bar{u}$ the following extension of v from $\mathbb{R} \setminus E_s$ to $\mathbb{R}^2_+ \setminus D_s$ in polar coordinates

$$\bar{u}(\rho, \theta) := \frac{\theta}{\pi^2}\alpha' + \left(1 - \frac{\theta}{\pi^2}\right)\beta', \quad \forall \theta \in [0, \pi), \ \forall \rho \geq s.$$

We construct the competitor w_ε simply gluing the function $\bar{u}$ and the function u_ε. Hence, consider the cut-off function φ in $C^\infty(\mathbb{R}^2_+)$, such that $\varphi \equiv 1$ in D_s, $\varphi \equiv 0$ in $\mathbb{R}^2_+ \setminus D_{s(\varepsilon)}$ and $|D\varphi| \leq 1/\varepsilon^{\frac{p-2}{2(p-1)}}$, where we denote

$$s(\varepsilon) := s + \varepsilon^{\frac{(p-2)}{2(p-1)}}.$$

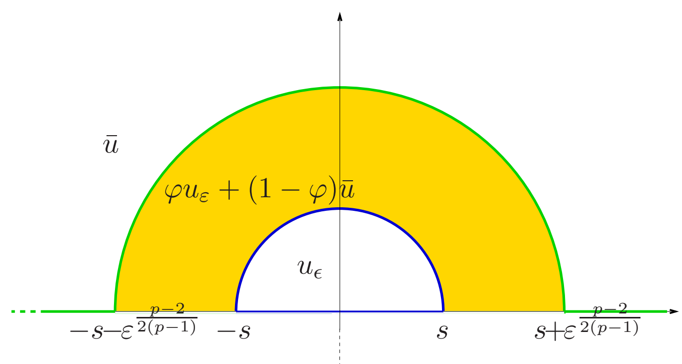

FIGURE 1. The competitor w_ε (see also [17, Fig. 2]).

Thus, we consider

$$w_\varepsilon := \begin{cases} u_\varepsilon & \text{in } D_s, \\ \varphi u_\varepsilon + (1-\varphi)\bar{u} & \text{in } D_{s(\varepsilon)} \setminus \ D_s, \\ \bar{u} & \text{in } \mathbb{R}^2_+ \setminus D_{s(\varepsilon)}. \end{cases}$$

Note that $\lim\limits_{t \to -\infty} Tw_\varepsilon(t) = \alpha'$ and $\lim\limits_{t \to +\infty} Tw_\varepsilon(t) = \beta'$.

Step 2: Choice of the annulus. We need to choose an annulus in the half-disk, in which we can recover a suitable quantity of energy of u_ε. Since u_ε has equi-bounded energy $H_\varepsilon(u_\varepsilon)$ in D_1, there exists $L > 0$ such that $\forall \varepsilon > 0 \ \exists s \in \left(\frac{1}{2}, 1 - \varepsilon^{\frac{p-2}{2(p-1)}}\right)$ such that

$$\varepsilon^{p-2} \int_{D_{s(\varepsilon)} \setminus D_s} |Du_\varepsilon|^p h^{2-p} dx + \frac{1}{\sqrt{\varepsilon}} \int_{E_{s(\varepsilon)} \setminus E_s} V(Tu_\varepsilon) d\mathcal{H}^1 \leq L\varepsilon^{\frac{p-2}{2(p-1)}}. \tag{4.11}$$

Step 3: Estimates. For every s like in Step 2, we have

$$H_\varepsilon(u_\varepsilon) \geq \varepsilon^{p-2} \int_{D_s \cap D_1^0} |Du_\varepsilon|^p h^{2-p} dx + \frac{1}{\sqrt{\varepsilon}} \int_{E_s} V(Tu_\varepsilon) d\mathcal{H}^1 \tag{4.12}$$

where D_1^0 is defined by (4.3).

By the scaling property of H_ε (see (4.4)) and noticing that here we have always $h^{2-p} = x_2^{2-p}$, we have

$$
\begin{aligned}
\varepsilon^{p-2}\int_{D_s\cap D_1^0}|Du_\varepsilon|^p x_2^{2-p}dx+ \frac{1}{\sqrt{\varepsilon}}\int_{E_s}V(Tu_\varepsilon)d\mathcal{H}^1 \\
&= \int_{(D_s\cap D_1^0)/\sqrt{\varepsilon}}|Du_\varepsilon^{(\varepsilon)}|^p y_2^{2-p}dy + \int_{E_s}V(Tw_\varepsilon^{(\varepsilon)})d\mathcal{H}^1 \\
&= H_1(w_\varepsilon^{(\varepsilon)},\mathbb{R}^2_+,\mathbb{R}) - \int_{(\mathbb{R}^2_+\setminus(D_s\cap D_1^0))/\sqrt{\varepsilon}}|Dw_\varepsilon^{(\varepsilon)}|^p y_2^{2-p}dy \\
&\quad - \int_{(\mathbb{R}\setminus E_s)/\sqrt{\varepsilon}}V(w_\varepsilon^{(\varepsilon)})d\mathcal{H}^1 \\
&\geq \gamma_p - \varepsilon^{p-2}\int_{\mathbb{R}^2_+\setminus(D_s\cap D_1^0)}|Dw_\varepsilon|^p x_2^{2-p}dx - \frac{1}{\sqrt{\varepsilon}}\int_{\mathbb{R}\setminus E_s}V(Tw_\varepsilon)d\mathcal{H}^1,
\end{aligned}
\tag{4.13}
$$

where we recall that $u^{(\varepsilon)}$ is the rescaled function defines by $u^{(\varepsilon)} = u(x/\sqrt{\varepsilon})$.

Using (4.12) and (4.13), we get

$$
\begin{aligned}
H_\varepsilon(u_\varepsilon) &\geq \gamma_p - \varepsilon^{p-2}\int_{\mathbb{R}^2_+\setminus(D_s\cap D_1^0)}|Dw_\varepsilon|^p x_2^{2-p}dx - \frac{1}{\sqrt{\varepsilon}}\int_{\mathbb{R}\setminus E_s}V(Tw_\varepsilon)d\mathcal{H}^1 \\
&\geq \gamma_p - \varepsilon^{p-2}\int_{\mathbb{R}^2_+\setminus(D_{s(\varepsilon)}\cap D_1^0)}|D\bar{u}|^p x_2^{2-p}dx - \varepsilon^{p-2}\int_{(D_{s(\varepsilon)}\setminus D_s)\cap D_1^0}|Dw_\varepsilon|^p x_2^{2-p}dx \\
&\quad -\frac{1}{\sqrt{\varepsilon}}\int_{\mathbb{R}\setminus E_s}V(Tw_\varepsilon)d\mathcal{H}^1.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\gamma_p &\leq H_\varepsilon(u_\varepsilon) + \varepsilon^{p-2}\int_{(\mathbb{R}^2_+\setminus D_{s(\varepsilon)})\cap D_1^0}|D\bar{u}|^p x_2^{2-p}dx \\
&\quad +\varepsilon^{p-2}\int_{(D_{s(\varepsilon)}\setminus D_s)\cap D_1^0}|Dw_\varepsilon|^p x_2^{2-p}dx + \frac{1}{\sqrt{\varepsilon}}\int_{\mathbb{R}\setminus E_s}V(Tw_\varepsilon)d\mathcal{H}^1.
\end{aligned}
\tag{4.14}
$$

Using the fact that $(X+Y+Z)^p \leq 3^{p-1}(X^p+Y^p+Z^p)$ and the definition of w_ε, the second integral in the right hand side of (4.14) can be estimated as follows

$$
\begin{aligned}
\varepsilon^{p-2}\int_{(D_{s(\varepsilon)}\setminus D_s)\cap D_1^0} |Dw_\varepsilon|^p x_2^{2-p}dx \;&\leq\; 3^{p-1}\varepsilon^{p-2}\int_{(D_{s(\varepsilon)}\setminus D_s)\cap D_1^0} |D\bar{u}|^p x_2^{2-p}dx \\
&\quad +3^{p-1}\varepsilon^{p-2}\int_{(D_{s(\varepsilon)}\setminus D_s)\cap D_1^0} |Du_\varepsilon|^p x_2^{2-p}dx \\
&\quad +3^{p-1}C_1(s+\varepsilon^{\frac{p-2}{2(p-1)}})^{4-p}\varepsilon^{\frac{(p-2)^2}{2(p-1)}}.
\end{aligned}
$$

Here we also used that $h(x_1,x_2)^{2-p} = x_2^{2-p}$ with $x_2 = \rho\sin\theta$, $\frac{s}{1+\sin\theta} \leq \rho \leq \frac{s(\varepsilon)}{1+\sin\theta}$ and it gives the estimate

$$
\begin{aligned}
\int_{(D_{s(\varepsilon)}\setminus D_s)\cap D_1^0} |D\varphi|^p|u_\varepsilon-\bar{u}|^p h^{2-p}dx \;&\leq\; \frac{(2m)^p}{\varepsilon^{\frac{p(p-2)}{2(p-1)}}}\int_0^\pi\int_s^{s(\varepsilon)} \rho^{2-p}\sin^{2-p}\theta\rho d\rho d\theta \\
&\leq\; \frac{(2m)^p s(\varepsilon)^{4-p}}{(4-p)\varepsilon^{\frac{p(p-2)}{2(p-1)}}}\int_0^\pi \frac{1}{\sin^{p-2}(\theta)}d\theta \\
&=\; \frac{C_1(s+\varepsilon^{\frac{p-2}{2(p-1)}})^{4-p}}{\varepsilon^{\frac{p(p-2)}{2(p-1)}}}.
\end{aligned}
$$

It follows

$$
\begin{aligned}
\gamma_p \;&\leq\; H_\varepsilon(u_\varepsilon) + 3^{p-1}\varepsilon^{p-2}\int_{\mathbb{R}^2_+\setminus D_s\cap D_1^0} |D\bar{u}|^p x_2^{2-p}dx + 3^{p-1}\varepsilon^{p-2}\int_{(D_{s(\varepsilon)}\setminus D_s)\cap D_1^0} |Du_\varepsilon|^p x_2^{2-p}dx \\
&\quad +\frac{1}{\sqrt{\varepsilon}}\int_{\mathbb{R}\setminus E_s} V(Tw_\varepsilon)d\mathcal{H}^1 + 3^{p-1}C_1(s+\varepsilon^{\frac{p-2}{2(p-1)}})^{4-p}\varepsilon^{\frac{(p-2)^2}{2(p-1)}}.
\end{aligned}
\tag{4.15}
$$

Using the definition of $\bar{u}$, we can compute the first integral

$$
\int_{\mathbb{R}^2_+\setminus D_s\cap D_1^0} |D\bar{u}|^p x_2^{2-p}dx \;=\; \frac{C_2}{s^{2(p-2)}}
$$

and (4.15) becomes

$$
\begin{aligned}
\gamma_p \;&\leq\; H_\varepsilon(u_\varepsilon) + 3^{p-1}\frac{C_2}{s^{2(p-2)}} + 3^{p-1}\varepsilon^{p-2}\int_{(D_{s(\varepsilon)}\setminus D_s)\cap D_1^0} |Du_\varepsilon|^p x_2^{2-p}dx \\
&\quad +\frac{1}{\sqrt{\varepsilon}}\int_{E_{s(\varepsilon)}\setminus E_s} V(Tw_\varepsilon)d\mathcal{H}^1 + 3^{p-1}C_1\varepsilon^{\frac{(p-2)^2}{2(p-1)}}.
\end{aligned}
\tag{4.16}
$$

Let us estimate the second integral in the right hand side of (4.16). Since $Tw_\varepsilon = \alpha'$ and $Tw_\varepsilon = \beta'$ on $\mathbb{R} \setminus E_{s(\varepsilon)}$, we have

$$\frac{1}{\sqrt{\varepsilon}} \int_{\mathbb{R}\setminus E_{s(\varepsilon)}} V(T\bar{u}) d\mathcal{H}^1 + \frac{1}{\sqrt{\varepsilon}} \int_{E_{s(\varepsilon)}\setminus E_s} V(Tw_\varepsilon) d\mathcal{H}^1 \equiv \frac{1}{\sqrt{\varepsilon}} \int_{E_{s(\varepsilon)}\setminus E_s} V(Tw_\varepsilon) d\mathcal{H}^1.$$

For every $\delta > 0$, let us define

$$E_\delta := \left\{ x \in E_{s(\varepsilon)} \setminus E_s : |Tu_\varepsilon - \beta'| > \delta \text{ and } |Tu_\varepsilon - \alpha'| > \delta \right\}.$$

Thanks to Step 2, there exists $N > \dfrac{L}{\omega_\delta}$ $\Big($where we denote by $\omega_\delta := \min\limits_{\substack{|t-\alpha'|\geq\delta \\ |t-\beta'|\geq\delta}} V(t)\Big)$ such that $\forall \delta > 0\ \exists \varepsilon_\delta$ such that

$$|E_\delta| \leq N \varepsilon^{\frac{p-2}{2(p-1)}} \sqrt{\varepsilon}, \quad \forall \varepsilon \leq \varepsilon_\delta. \tag{4.17}$$

In particular, choosing δ small, the convexity of V near its wells provides

$$\begin{aligned} \frac{1}{\sqrt{\varepsilon}} \int_{(E_{s(\varepsilon)}\setminus E_s)\setminus E_\delta} V(Tw_\varepsilon) d\mathcal{H}^1 + \frac{1}{\sqrt{\varepsilon}} \int_{E_\delta} V(Tw_\varepsilon) d\mathcal{H}^1 \ &\leq\ \frac{1}{\sqrt{\varepsilon}} \int_{(E_{s(\varepsilon)}\setminus E_s)\setminus E_\delta} V(Tu_\varepsilon) d\mathcal{H}^1 \\ &\quad + \omega_m N \varepsilon^{\frac{p-2}{2(p-1)}}, \end{aligned} \tag{4.18}$$

where $\omega_m := \max\limits_{|t|<m} V(t)$ and we used the inequality (4.17).

Finally, by (4.11), (4.16) and (4.18), we obtain, for every $\delta > 0$

$$\begin{aligned} H_\varepsilon(u_\varepsilon) \ &\geq\ \gamma_p - \Bigg(3^{p-1} \left(\varepsilon^{p-2} \int_{D_{s(\varepsilon)}\setminus D_s} |Du_\varepsilon|^p h^{2-p} dx + \frac{1}{\sqrt{\varepsilon}} \int_{E_{s(\varepsilon)}\setminus E_s} V(Tu_\varepsilon) d\mathcal{H}^1 \right) \\ &\qquad 3^{p-1} \frac{C_2}{s^{2(p-2)}} \varepsilon^{p-2} + 3^{p-1} C_1 (s + \varepsilon^{\frac{p-2}{2(p-1)}})^{4-p} \varepsilon^{\frac{(p-2)^2}{2(p-1)}} + \omega_m N \varepsilon^{\frac{p-2}{2(p-1)}} \Bigg) \\ &\geq\ \gamma_p - \Bigg(3^{p-1} L \varepsilon^{\frac{p-2}{2(p-1)}} + 3^{p-1} \frac{C_2}{s^{2(p-2)}} \varepsilon^{p-2} + 3^{p-1} C_1 (s + \varepsilon^{\frac{p-2}{2(p-1)}})^{4-p} \varepsilon^{\frac{(p-2)^2}{2(p-1)}} \\ &\qquad + \omega_m N \varepsilon^{\frac{p-2}{2(p-1)}} \Bigg). \end{aligned}$$

Notice that for every $\varepsilon > 0$, $s \in \left(1/2, 1 - \varepsilon^{\frac{p-2}{2(p-1)}}\right)$. Hence, taking the limit as $\varepsilon \to 0$, we get $\liminf\limits_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon) \geq \gamma_p$, which concludes the proof in the case of a function v with one jump, i.e. $\mathcal{H}^0(Sv) = 1$. The case $\mathcal{H}^0(Sv) > 1$ can be treated similarly. □

4.3. **Reduction to the flat case.** According to the idea of Alberti, Bouchitte and Seppecher in [3], we will prove the Theorem 2.1 after arguments of slicing and blow-up. More precisely, it is possible to deform each neighborhood in a bi-Lipschitz fashion in order to straighten the boundary of Ω, without changing much the functional (see [3, Proposition 4.9] and [17, Proposition 5.2]).

To this aim, we recall the definition of the "isometry defect", introduced by Alberti, Bouchitté and Seppecher[3].

As usual, we denote by $O(3)$ the set of linear isometries on $\mathbb{R}^3$.

Definition 4.4. *Let $A_1, A_2 \subset \mathbb{R}^3$ and let $\Psi : \overline{A_1} \to \overline{A_2}$ bi-Lipschitz homeomorphism. Then the* "isometry defect $\delta(\Psi)$ of Ψ" *is the smallest constant δ such that*

$$\operatorname{dist}(D\Psi(x), O(3)) \leq \delta, \quad \textit{for a.e. } x \in A_1. \tag{4.19}$$

Here $D\Psi(x)$ is regarded as a linear mapping of $\mathbb{R}^3$ into $\mathbb{R}^3$. The distance between linear mappings is induced by the norm $\|\cdot\|$, which, for every L, is defined as the supremum of $|Lv|$ over all v such that $|v| \leq 1$. Hence, for every $L_1, L_2 : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\operatorname{dist}(L_1, L_2) := \sup_{x:|x|\leq 1} |L_1(x) - L_2(x)|.$$

By (4.19), we get

$$\|D\Psi(x)\| \leq 1 + \delta(\Psi) \text{ for a.e. } x \in A_1, \tag{4.20}$$

and then Ψ is $(1+\delta(\Psi))$-Lipschitz continuous on every convex subset of A_1. Similarly, Ψ^{-1} is $(1-\delta(\Psi))^{-1}$-Lipschitz continuous on every convex subset of A_2.

The following proposition shows that the localized energy $F_\varepsilon(u, B_r(x)\cap\Omega, B_r(x)\cap\partial\Omega)$ can be replaced by the energy $F_\varepsilon(u, D_r, E_r)$, where B_r is the two-dimensional ball of radius r centered in the origin.

Proposition 4.5. *For every $x \in \partial\Omega$ and every positive r smaller than a certain critical value $r_x > 0$, there exists a bi-Lipschitz map $\Psi_r : \overline{D_r} \to \overline{\Omega \cap B_r(x)}$ such that*

(a) *Ψ_r takes D_r onto $\Omega \cap B_r(x)$ and E_r onto $\partial\Omega \cap B_r(x)$;*
(b) *Ψ_r is of class C^1 on D_r and $\|D\Psi_r - I\| \leq \delta_r$ everywhere in D_r, where $\delta_r \to 0$ as $r \to 0$.*

In particular, the isometry defect of Ψ_r vanishes as $r \to 0$. Moreover,

$$F_\varepsilon(u, B_r(x)\cap\Omega, B_r(x)\cap\partial\Omega) \geq (1-\delta(\Psi))^{p+3} F_\varepsilon(u\circ\Psi, D_r, E_r).$$

The proof is a simple modification of the one by Alberti, Bouchitté and Seppecher in [3], Proposition 4.9 and Proposition 4.10, where they treat the case $p = 2$ (see also [9, Proposition 6.1]).

Finally, we need to prove compactness and a lower bound inequality for the following energies

$$F_\varepsilon(u,\mathcal{D},\mathcal{E}) = \varepsilon^{p-2}\int_{\mathcal{D}} |Du|^p h^{2-p} dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_{\mathcal{D}} W(u) h^{\frac{p-2}{p-1}} dx + \frac{1}{\sqrt{\varepsilon}}\int_{\mathcal{E}} V(Tu) d\mathcal{H}^2,$$

where $\mathcal{D} \subset \mathbb{R}^3$ is the open half-ball centered in 0 with radius $r > 0$ and $\mathcal{E} \subset \mathbb{R}^2$ is defined by

$$\mathcal{E} := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x| \leq r, x_3 = 0\right\}.$$

We will reduce to Proposition 4.2 and Proposition 4.3 via a suitable slicing argument.

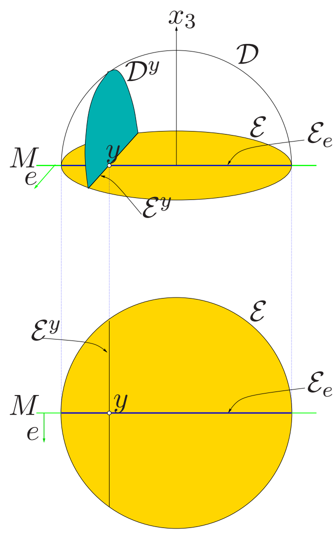

FIGURE 2. The sets $\mathcal{D}, \mathcal{E}, \mathcal{E}_e, \mathcal{E}^y$ and $\mathcal{D}^y$ (see also [3, Fig. 4]).

We use the following notation: e is a unit vector in the plane $P := \{x_3 = 0\}$; M is the orthogonal complement of e in P; π is the projection of $\mathbb{R}^3$ onto M; for every $y \in \mathcal{E}_e := \pi(\mathcal{E})$, we denote by $\mathcal{E}^y := \pi^{-1}(y) \cap \mathcal{E}$, $\mathcal{D}^y := \pi^{-1}(y) \cap \mathcal{D}$ (see Fig. 2); for every function u defined on $\mathcal{D}$ we consider the trace of u on $\mathcal{E}^y$, i.e., the one-dimensional function

$$u_e^y(t) := u(y + te).$$

Proposition 4.6. *Let $(u_\varepsilon) \subset W^{1,p}(\mathcal{D}, h^{2-p}; [-m, m])$ be a sequence with uniformly bounded energies $F_\varepsilon(u_\varepsilon, \mathcal{D}, \mathcal{E})$. Then the traces Tu_ε are pre-compact in $L^1(\mathcal{E})$ and every cluster point belongs to $BV(\mathcal{E}; \{\alpha', \beta'\})$. Moreover, if $Tu_\varepsilon \to v$ in $L^1(\mathcal{E})$, then*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \mathcal{D}, \mathcal{E}) \geq \gamma_p \left|\int_{\mathcal{E}\cap Sv} \nu_v\right| d\mathcal{H}^1. \tag{4.21}$$

Proof. By Fubini's Theorem, for every $\varepsilon > 0$, we get

$$
\begin{aligned}
F_\varepsilon(u_\varepsilon, \mathcal{D}, \mathcal{E}) &\geq \varepsilon^{p-2} \int_{\mathcal{D}} |Du_\varepsilon|^p h^{2-p} dx + \frac{1}{\sqrt{\varepsilon}} \int_{\mathcal{E}} V(Tu_\varepsilon) d\mathcal{H}^2 \\
&\geq \int_{\mathcal{E}_e} \left[\varepsilon^{p-2} \int_{\mathcal{D}^y} |Du_\varepsilon^y|^p h^{2-p} dx + \frac{1}{\sqrt{\varepsilon}} \int_{\mathcal{E}^y} V(Tu_\varepsilon^y) d\mathcal{H}^1 \right] dy \\
&= \int_{\mathcal{E}_e} H_\varepsilon(u_\varepsilon^y, \mathcal{D}^y, \mathcal{E}^y) dy
\end{aligned}
\tag{4.22}
$$

and the 2D functional in the integration above has been studied in the last section. The remaining part of the proof follows exactly as in [3, Proposition 4.7]. □

4.4. **Existence of an optimal profile.** We conclude this section with the proof of the existence of a minimum for the optimal profile problem (4.5), showing that the minimum for γ_p is achieved by a function with non-decreasing trace.

Proposition 4.7. *The minimum for γ_p defined by (4.5) is achieved by a function u such that Tu is a non-decreasing function in $\mathbb{R}$.*

Proof. Note that, since the energy H_1 is decreasing under truncation by α' and β', it is not restrictive to minimize the problem (4.5) with the additional condition $\alpha' \leq u \leq \beta'$.

We denote by

$$
X := \left\{ w : \mathbb{R}^2_+ \to [\alpha', \beta'] : w \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+), \lim_{t \to -\infty} Tw(t) = \alpha', \lim_{t \to +\infty} Tw(t) = \beta' \right\}
$$

$$
X^* := \left\{ w \in X : Tw \text{ is non-decreasing}, \ Tw(t) \geq \frac{\alpha' + \beta'}{2} \text{ for } t > 0, \ Tw(t) \leq \frac{\alpha' + \beta'}{2} \text{ for } t < 0 \right\}.
$$

Let u be in X, we denote by $u^\star$ its monotone increasing rearrangement in direction x_1. Since monotone increasing rearrangement in one direction decreases the weighted L^p-norm of the gradient (see [5, Theorem 3]), the infimum of H_1 on X is equal to the infimum of H_1 on X^*.

Now we can prove by Direct Method that the infimum of H_1 on X^* is achieved.

Take a minimizing sequence $(u_n) \subset X^*$. In particular, $H_1(u_n, \mathbb{R}^2_+, \mathbb{R}) \leq C$, Du_n converges weakly to Du in $L^p(\mathbb{R}^2_+, h^{2-p})$ and u_n converges to u weakly in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2_+, h^{2-p})$. Since $\displaystyle\int_{\mathbb{R}^2_+} |Du_n|^p h^{2-p} dx$ is bounded, we can find a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+, h^{2-p})$ such that (up to a subsequence)

$$
Du_n \rightharpoonup Du \text{ in } L^p(\mathbb{R}^2_+, h^{2-p}) \text{ and } u_n \rightharpoonup u \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^2_+, h^{2-p}).
$$

By the trace embedding of $W^{1,p}(\mathbb{R}^2_+, h^{2-p})$ in $W^{2-3/p,p}(\mathbb{R})$, we have

$$Tu_n \rightharpoonup Tu \text{ in } W^{2-3/p,p}_{\mathrm{loc}}(\mathbb{R}).$$

By the compact embedding of $C^0_{\mathrm{loc}}(\mathbb{R})$ in $W^{2-3/p,p}_{\mathrm{loc}}(\mathbb{R})$ (see [1, Theorem 7.34]), we have that, up to a subsequence, Tu_n locally uniformly converges to Tu. Thus Tu is non-decreasing and satisfies

$$Tu(t) \geq \frac{\alpha' + \beta'}{2} \text{ for } t > 0 \quad \text{and} \quad Tu(t) \leq \frac{\alpha' + \beta'}{2} \text{ for } t < 0.$$

Let us show that $\lim_{t\to-\infty} Tu(t) = \alpha'$ and $\lim_{t\to+\infty} Tu(t) = \beta'$. Since Tu is non-decreasing in $[\alpha', \beta']$, there exist $a \leq \frac{\alpha' + \beta'}{2}$ and $b \geq \frac{\alpha' + \beta'}{2}$ such that

$$a := \lim_{t\to-\infty} Tu(t) \text{ and } b := \lim_{t\to+\infty} Tu(t).$$

By contradiction, we assume that either $a \neq \alpha'$ or $b \neq \beta'$. Then, since V is continuous and strictly positive in (α', β'), we obtain

$$\int_{\mathbb{R}} V(Tu)d\mathcal{H}^1 = +\infty,$$

This is impossible, because, by Fatou's Lemma, we have

$$\int_{\mathbb{R}} V(Tu)d\mathcal{H}^1 \leq \liminf_{n\to+\infty} \int_{\mathbb{R}} V(Tu_n)d\mathcal{H}^1 < \liminf_{n\to+\infty} H_1(u_n, \mathbb{R}^2_+, \mathbb{R}) < +\infty.$$

Hence, u is in X^*. Since H_1 is clearly lower semicontinuous on sequences such that $Du_n \rightharpoonup Du$ in $L^p(\mathbb{R}^2_+, h^{2-p})$ and $Tu_n \to Tu$ pointwise, this concludes the proof. □

5. Proof of the main result

In the previous sections, we have obtained the main ingredients of the proof of Theorem 2.1, which follows as in the quadratic case in [3], with the needed modifications due to the presence of the weight (like in [9]) and to the super-quadratic growth in the singular perturbation term (like in [17]).

5.1. **Compactness.** Let a sequence $(u_\varepsilon) \subset W^{1,p}(\Omega)$ be given such that $F_\varepsilon(u_\varepsilon)$ is bounded. Since $F_\varepsilon(u_\varepsilon) \geq F_\varepsilon(u_\varepsilon, \Omega, \emptyset) \equiv G_\varepsilon(u_\varepsilon, \Omega)$, by the statement (i) of Theorem 3.1, the sequence (u_ε) is pre-compact in $L^1(\Omega)$ and there exists $u \in BV(\Omega; \{\alpha, \beta\})$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$.

On the boundary, by slicing, we may use Proposition 4.6, that implies that (Tu_ε) is pre-compact in $L^1(\partial\Omega)$ and that its cluster points are in $BV(\partial\Omega; \{\alpha', \beta'\})$. □

5.2. **Lower bound inequality.** The proof of the lower bound inequality of Theorem 2.1 follows by putting together the results in the interior, Theorem 3.1-(ii) and Proposition 3.2-(i), and the ones on the boundary, Section 4 and Proposition 4.3.

Let a sequence $(u_\varepsilon) \subset W^{1,p}(\Omega, h^{2-p})$ be given such that $u_\varepsilon \to u \in BV(\Omega, \{\alpha, \beta\})$ in $L^1(\Omega)$ and $Tu_\varepsilon \to v \in BV(\partial\Omega, \{\alpha'\beta'\})$ in $L^1(\partial\Omega)$. We have to prove that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v), \tag{5.1}$$

where Φ is given by (2.2).
Clearly, we can assume that $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < +\infty$.

For every $\varepsilon > 0$, let μ_ε be the energy distribution associated with F_ε with configuration u_ε; i.e., μ_ε is the positive measure given by

$$\mu_\varepsilon(B) := \varepsilon^{p-2} \int_{\Omega\cap B} |Du_\varepsilon|^p dx + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}} \int_{\Omega\cap B} W(u_\varepsilon)dx + \frac{1}{\sqrt{\varepsilon}} \int_{\partial\Omega\cap B} V(Tu_\varepsilon)d\mathcal{H}^2, \tag{5.2}$$

for every $B \subset \mathbb{R}^3$ Borel set.
Similarly, let us define

$$\begin{aligned} \mu^1(B) &:= \sigma_p \mathcal{H}^2(Su \cap B), \\ \mu^2(B) &:= c_p \int_{\partial\Omega\cap B} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2, \\ \mu^3(B) &:= \gamma_p \mathcal{H}^1(Sv \cap B). \end{aligned}$$

The total variation $\|\mu_\varepsilon\|$ of the measure μ_ε is equal to $F_\varepsilon(u_\varepsilon)$, and $\|\mu^1\| + \|\mu^2\| + \|\mu^3\|$ is equal to $\Phi(u, v)$. The quantity $\|\mu_\varepsilon\|$ is bounded and we can assume that μ_ε converges in the sense of measures to some finite measure μ. Then, by the lower semicontinuity of the total variation, we have

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \equiv \liminf_{\varepsilon\to 0} \|\mu_\varepsilon\| \geq \|\mu\|.$$

Since the measures μ^i are mutually singular, we obtain the lower bound inequality (5.1) if we prove that

$$\mu \geq \mu^i, \quad \text{for } i = 1, 2, 3. \tag{5.3}$$

It is enough to prove that $\mu(B) \geq \mu^i(B)$ for all sets $B \subset \mathbb{R}^3$ such that $B \cap \Omega$ is a Lipschitz domain and $\mu(\partial B) = 0$.

By the inequality of statement (ii) of Theorem 3.1, we have

$$\mu(B) = \lim_{\varepsilon\to 0} \mu_\varepsilon(B) \geq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega \cap B, \emptyset) \geq \sigma_p \mathcal{H}^2(Su \cap B) \equiv \mu^1(B).$$

Similarly, we can prove that $\mu \geq \mu^2$. We have

$$\begin{aligned} \mu(B) &= \lim_{\varepsilon\to 0} \mu_\varepsilon(B) \geq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega\cap B, \emptyset) \\ &\geq c_p \int_{\partial\Omega\cap B} |\mathcal{W}(Tu) - \mathcal{W}(v)| d\mathcal{H}^2 \equiv \mu^2(B), \end{aligned}$$

where we used Proposition 3.2-(i) with $A := B\cap\Omega$ and $A' := B\cap\partial\Omega$.

The inequality $\mu \geq \mu^3$ requires a different argument. Notice that μ^3 is the restriction of $\mathcal{H}^1$ to the set Sv, multiplied by the factor γ_p. Thus, if we prove that

$$\liminf_{r\to 0} \frac{\mu(B_r(x))}{2r} \geq \gamma_p, \quad \mathcal{H}^1\text{-a.e. } x\in Sv, \tag{5.4}$$

for $B_r(x)$ as in Proposition 4.5, we obtain the required inequality.

Let us fix $x \in Sv$ such that there exists $\lim_{r\to 0} \frac{\mu(B_r(x))}{2r}$ and Sv has one-dimensional density equal to 1. We denote by ν_v the unit normal at x.

For r small enough, we choose a map Ψ_r such as in Proposition 4.5. Thus we have $\Psi_r(\overline{D_r}) = \Omega\cap B_r(x)$, $\Psi_r(E_r) = \partial\Omega\cap B_r(x)$ and $\delta(\Psi_r) \to 0$ as $r\to 0$.

Let us set

$$\bar{u}_\varepsilon := u_\varepsilon \circ \Psi_r \text{ and } \bar{v} := v\circ\Psi_r.$$

Hence, $T\bar{u}_\varepsilon \to \bar{v}$ in $L^1(E_r)$ and $\bar{v} \in BV(E_r, \{\alpha', \beta'\})$. So, thanks to Proposition 4.5, we obtain

$$\begin{aligned} \mu(B_r(x)) &= \lim_{\varepsilon\to 0} \mu_\varepsilon(B_r(x)) \\ &= \lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega\cap B_r(x), \partial\Omega\cap B_r(x)) \\ &\geq \liminf_{\varepsilon\to 0} (1-\delta(\Psi_r))^{p+3} F_\varepsilon(\bar{u}_\varepsilon, D_r, E_r). \end{aligned} \tag{5.5}$$

Moreover, by Proposition 4.6, we have

$$\liminf_{\varepsilon\to 0} F_\varepsilon(\bar{u}_\varepsilon, D_r, E_r) \geq \gamma_p \left| \int_{S\bar{v}\cap E_r} \nu_v d\mathcal{H}^1 \right|. \tag{5.6}$$

Finally, we notice that $\delta(\Psi_r)$ vanishes and $\left| \int_{S\bar{v}\cap E_r} \nu_v d\mathcal{H}^1 \right| = 2r + o(r)$ as r goes to 0. So (5.5) and (5.6) give the following inequality

$$\frac{\mu(B_r(x))}{2r} \geq \gamma_p \left(1 + \frac{o(r)}{2r}\right) \text{ as } r\to 0,$$

that implies $\mu \geq \mu^3$. This concludes the proof of the lower bound inequality. □

5.3. **Upper bound inequality.** We will construct an optimal sequence (u_ε) according to Theorem 2.1-(iii) in a suitable partition of Ω, as in [3, Theorem 2.6-(iii)], but for the estimate of the boundary effect we will use the optimal profile problem (4.5) in connection with the results proved in the previous section. We do not present all the details, just sketch the main ideas and state the needed lemmas.

Fix $(u,v) \in BV(\Omega;\{\alpha',\beta'\}) \times BV(\partial\Omega;\{\alpha',\beta'\})$. It is not restrictive to assume that the singular sets Su and Sv are closed manifolds of class C^2 without boundary (see [10, Theorem 1.24]). We may also assume that u and v (up to modifications on negligible sets) are constant in each connected component of $\Omega \setminus Su$ and $\partial\Omega \setminus Sv$, respectively.

The idea is to construct a *partition* of Ω in four subsets, and to use the preliminary convergence results of the previous sections to obtain the upper bound inequality.

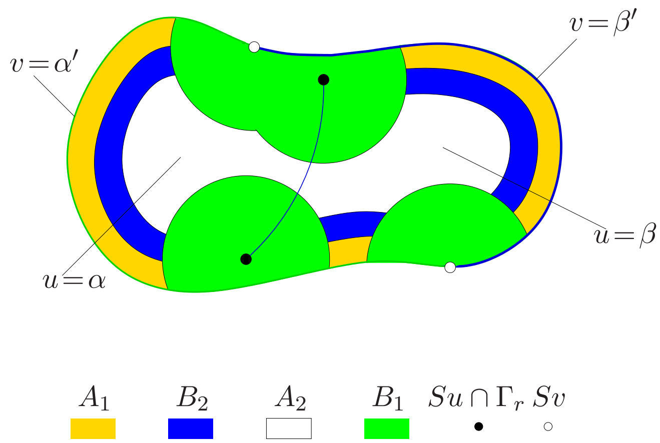

FIGURE 3. Upper bound inequality - partition of Ω (see [3, Fig. 6]).

For every $r > 0$, we set

$$\Gamma_r := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) = r\}.$$

Step 1 : Construction of the partition.

Fix $r > 0$ such that Γ_r and Γ_{2r} are Lipschitz surfaces and $Su \cap \Gamma_r$ is a Lipschitz curve.

Now, we are ready to construct the following partition of Ω:

$$\begin{aligned} B_1 &:= \{x \in \Omega : \operatorname{dist}(x, Sv \cup (Su \cap \Gamma_r)) < 3r\}, \\ A_1 &:= \left\{x \in \Omega \setminus \overline{B}_1 : \operatorname{dist}(x, \partial\Omega) < r\right\}, \\ B_2 &:= \left\{x \in \Omega \setminus \overline{B}_1 : r < \operatorname{dist}(x, \partial\Omega) < 2r\right\}, \\ A_2 &:= \left\{x \in \Omega \setminus \overline{B}_1 : \operatorname{dist}(x, \partial\Omega) > 2r\right\}. \end{aligned}$$

(See Fig. 3)

For every $r > 0$ and every $\varepsilon < r^{\frac{p-1}{p-2}}$ we construct a Lipschitz function $u_\varepsilon = u_{\varepsilon,r}$ in each subset, with controlled Lipschitz constant.

Step 2 : Construction of $u_{\varepsilon,r}$ in A_2.
In A_2, we take u_ε being the optimal sequence for the functional G_ε in the set A_2 as in Theorem 3.1-(iii) and we extend it to ∂A_2 by continuity. Note that u_ε converges to u pointwise in A_2 and uniformly on $\partial A_2 \cap \partial B_2$, and

$$\begin{aligned} F_\varepsilon(u_\varepsilon, A_2, \emptyset) \equiv G_\varepsilon(u_\varepsilon, A_2) &\leq \sigma_p \mathcal{H}^2(Su \cap A_2) + o(1) \\ &\leq \sigma_p \mathcal{H}^2(Su \cap A_2) + o(1), \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{5.7}$$

Step 3 : Construction of $u_{\varepsilon,r}$ in A_1.
The function u is constant (equal to α or β) on every connected component A of A_1, and the function v is constant (equal to α' or β') on $\partial A \cap \partial\Omega$. We can extend it to ∂A_1 with continuity; Proposition 3.2-(ii) gives

$$\begin{aligned} F_\varepsilon(u_\varepsilon, A_1, \partial A_1 \cap \partial\Omega) \equiv G_\varepsilon(u_\varepsilon, A_1) &\leq c_p \int_{\partial A_1 \cap \partial\Omega} |\mathcal{W}(Tu(x)) - \mathcal{W}(v(x))| d\mathcal{H}^2 \\ &\quad + o(1), \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{5.8}$$

Step 4 : Construction of $u_{\varepsilon,r}$ in B_2.
Following [3], to construct u_ε on B_2, we need to "glue" the values of A_1 and A_2. Take a cut-off function ξ such that $\xi = 1$ in A_1 and $\xi = 0$ in A_2 and consider the function

$$u_\varepsilon = \xi \bar{u}_1 + (1 - \xi)\bar{u}_2,$$

where $\bar{u}_i$ is the extension to B_2 of $u_\varepsilon|_{A_i}$. Then, when $\varepsilon \to 0$, by the decay of the function ξ, we have

$$\begin{aligned} \varepsilon^{p-2} \int_{B_2} |Du_\varepsilon|^p h^{2-p} dx \leq C \Bigg(&\int_{B_2} |D\bar{u}_1|^p h^{2-p} dx + \int_{B_2} |D\bar{u}_2|^p h^{2-p} dx \\ &+ \int_{B_2} |D\xi|^p |\bar{u}_1 - \bar{u}_2|^p h^{2-p} dx \Bigg) = o(1). \end{aligned}$$

Step 5 : Construction of $u_{\varepsilon,r}$ in B_2.
Finally, for the last part B_1, we will use an optimal profile for the minimum problem (4.5).

By Proposition 4.7, there exists $\psi \in L^1_{\rm loc}(\mathbb{R}^2_+)$ such that $T\psi(t) \to \alpha'$ as $t \to -\infty$, $T\psi(t) \to \beta'$ as $t \to +\infty$ and $H_1(\psi, \mathbb{R}^2_+, \mathbb{R}) = \gamma_p$. We can construct a function $w_\varepsilon : \mathbb{R}^2_+ \to \mathbb{R}$ following the method used to provide a good competitor u_δ in the proof of Proposition 4.3.

For every $\varepsilon > 0$, $\rho_\varepsilon, \sigma_\varepsilon \in \mathbb{R}$, we take a cut-off function $\xi \in C^\infty(\mathbb{R}^2_+)$ such that $\xi \equiv 1$ on $(\mathbb{R}^2_+) \setminus D_{\rho_\varepsilon}$ and $\xi \equiv 0$ on D_{σ_ε} such that $|D\xi| \le \frac{1}{|\rho_\varepsilon - \sigma_\varepsilon|}$. We denote by $\bar u$ the function defined in polar coordinates $\theta \in [0, \pi)$, $\rho \in [0, +\infty)$ as follows

$$\bar u(\theta, \rho) := \frac{\theta}{\pi}\alpha' + \left(1 - \frac{\theta}{\pi}\right)\beta'.$$

We define w_ε as

$$w_\varepsilon(x) := \begin{cases} \psi(\frac{x}{\varepsilon}) & \text{if } x \in D_{\sigma_\varepsilon}, \\ \\ \xi(x)\bar u(x) + (1 - \xi(x))\psi(\frac{x}{\varepsilon}) & \text{if } x \in D_{\rho_\varepsilon} \setminus D_{\sigma_\varepsilon}, \\ \\ \bar u(x) & \text{if } x \in \mathbb{R}^2_+ \setminus D_{\rho_\varepsilon}. \end{cases}$$

Let us show that we can choose ρ_ε and σ_ε such that w_ε satisfies the following inequality

$$\varepsilon^{p-2}\int_{D_{\rho_\varepsilon}} |Dw_\varepsilon|^p h^{2-p} dx + \frac{1}{\sqrt{\varepsilon}}\int_{E_{\rho_\varepsilon}} V(Tw_\varepsilon) \le \gamma_p + o(1), \text{ as } \varepsilon \to 0. \tag{5.9}$$

By the definition of w_ε and by standard changing variable formula ($y = x/\sqrt{\varepsilon}$), we have

$$\begin{aligned} \varepsilon^{p-2}\int_{D_{\rho_\varepsilon}} |Dw_\varepsilon|^p h^{2-p} dx &= \int_{(D_{\rho_\varepsilon} \cap D^0_1)/\sqrt{\varepsilon}} |Dw^{(\varepsilon)}_\varepsilon|^p y_2^{2-p} dy \\ &\quad + \varepsilon^{p-2}\int_{D_{\rho_\varepsilon} \cap (D^0_1)^c} |Dw_\varepsilon|^p \left(1 - \sqrt{x_1^2 + x_2^2}\right)^{2-p} dx \\ &\le \int_{\mathbb{R}^2_+} |D\psi|^p y_2^{2-p} dy + \varepsilon^{p-2}\int_{(D_{\rho_\varepsilon} \setminus D_{\sigma_\varepsilon}) \cap D^0_1} |Dw_\varepsilon|^p x_2^{2-p} dx \\ &\quad + \varepsilon^{p-2}\int_{D_{\rho_\varepsilon} \cap (D^0_1)^c} |Dw_\varepsilon|^p \left(1 - \sqrt{x_1^2 + x_2^2}\right)^{2-p} dx \\ &=: \int_{\mathbb{R}^2_+} |D\psi|^p y_2^{2-p} dy + I_1 + I_2, \end{aligned} \tag{5.10}$$

where, D^1_0 is defined by (4.3).

Notice that when $\rho_\varepsilon << 1$, the integral I_2 is zero, since $D_{\rho_\varepsilon} \cap (D^0_1)^c$ is empty, hence we have

$$\begin{aligned} H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) &\le H_1(\psi, \mathbb{R}^2_+, \mathbb{R}) + I_1 \\ &= \gamma_p + I_1. \end{aligned} \tag{5.11}$$

Thus, to obtain (5.9), it suffices to estimate the integral I_1. We may work more or less like in the proof of Proposition 4.3.

We have

$$\begin{aligned} I_1 \;&\leq\; 3^{p-1}\varepsilon^{p-2}\int_{(D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon})\cap D_1^0}|D\psi(\frac{x}{\varepsilon})|^p x_2^{2-p}dx + 3^{p-1}\varepsilon^{p-2}\int_{(D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon})\cap D_1^0}|D\bar{u}|^p x_2^{2-p}dx \\ &\quad +3^{p-1}\varepsilon^{p-2}\int_{(D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon})\cap D_1^0}|D\xi|^p|\psi(\frac{x}{\varepsilon})-\bar{u}|^p x_2^{2-p}dx \end{aligned} \tag{5.12}$$

and the last two integrals in the right part of (5.12) can be explicitly estimated as follows

$$\begin{aligned} 3^{p-1}\varepsilon^{p-2}\int_{(D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon})\cap D_1^0}|D\bar{u}|^p x_2^{2-p}dx \;&=\; 3^{p-1}\varepsilon^{p-2}\frac{|\beta'-\alpha'|^p}{\pi^p}\int_0^\pi\int_{\sigma_\varepsilon}^{\rho_\varepsilon}\frac{(\rho\sin\theta)^{2-p}}{\rho^p}\rho d\rho d\theta \\ &\leq\; C_1\frac{\varepsilon^{p-2}}{{\rho_\varepsilon}^{2(p-2)}} \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} 3^{p-1}\varepsilon^{p-2}\int_{D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon}\cap(D_1^0)^c}|D\xi|^p|\psi(\frac{x}{\varepsilon})-\bar{u}|^p x_2^{2-p}dx \;&\leq\; 3^{p-1}\varepsilon^{p-2}\frac{(2m)^p}{(\rho_\varepsilon-\sigma_\varepsilon)^p}\int_0^\pi\int_{\sigma_\varepsilon}^{\rho_\varepsilon}\rho(\rho\sin\theta)^{2-p}d\theta d\rho \\ &\leq\; C_2\frac{\varepsilon^{p-2}\rho_\varepsilon^{4-p}}{(\rho_\varepsilon-\sigma_\varepsilon)^p}. \end{aligned} \tag{5.14}$$

Finally, by (5.12), (5.13) and (5.14), the inequality (5.11) becomes

$$\begin{aligned} H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) \;&\leq\; \gamma_p + 3^{p-1}\varepsilon^{p-2}\int_{(D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon})\cap D_1^0}|D\psi(\frac{x}{\varepsilon})|^p x_2^{2-p}dx \\ &\quad +C_1\frac{\varepsilon^{p-2}}{\rho_\varepsilon^{2(p-2)}} + C_2\frac{\varepsilon^{p-2}\rho_\varepsilon^{4-p}}{(\rho_\varepsilon-\sigma_\varepsilon)^p} \\ &\leq\; \gamma_p + o(1) \quad \text{as } \varepsilon\to 0, \end{aligned} \tag{5.15}$$

where we also used that, since $\int_{\mathbb{R}^2_+}|D\psi|^p y_2^{2-p}$ is finite, by suitable choosing ρ_ε and σ_ε we get

$$\begin{aligned} 3^{p-1}\varepsilon^{p-2}\int_{(D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon})\cap D_1^0}|D\psi(\frac{x}{\varepsilon})|^p x_2^{2-p}dx \;&=\; 3^{p-1}\int_{((D_{\rho_\varepsilon}\setminus D_{\sigma_\varepsilon})\cap D_1^0)/\sqrt{\varepsilon}}|D\psi|^p y_2^{2-p}dy \\ &=\; o(1) \quad \text{as } \varepsilon\to 0. \end{aligned} \tag{5.16}$$

Since the neighborhood B_1 is Lipschitz equivalent (modulo some multiplicative constant) to the product $Sv \times D_{\rho_\varepsilon}$, we can construct the following transplanted function $\bar{w}_\varepsilon$

$$\bar{w}_\varepsilon(x,z) := w_\varepsilon(x) \quad \forall x \in Sv, \forall z \in \mathbb{R}^2_+.$$

By Fubini's Theorem, we obtain

$$\begin{aligned} F_\varepsilon(\bar{w}_\varepsilon, Sv \times D_{\rho_\varepsilon}, Sv \times E_{\rho_\varepsilon}) \ &= \ \mathcal{H}^1(Sv)\left(H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) + \frac{1}{\varepsilon^{\frac{p-2}{p-1}}}\int_{D_{\rho_\varepsilon}} W(w_\varepsilon) h^{\frac{p-2}{p-1}} dx\right) \\ &\leq \ \mathcal{H}^1(Sv)\left(H_\varepsilon(w_\varepsilon, D_{\rho_\varepsilon}, E_{\rho_\varepsilon}) + C_3 \frac{\rho_\varepsilon^2}{\varepsilon^{\frac{p-2}{p-1}}}\right). \end{aligned} \tag{5.17}$$

Hence, by suitably choosing ρ_ε and σ_ε(i.e., such that $\frac{\varepsilon^{p-2}}{\rho_\varepsilon^{2(p-2)}}$, $\frac{\varepsilon^{p-2}\rho_\varepsilon^{4-p}}{(\rho_\varepsilon-\sigma_\varepsilon)^p}$ and $\frac{\rho_\varepsilon^2}{\varepsilon^{\frac{p-2}{p-1}}} \to 0$ as $\varepsilon \to 0$ and (5.16) holds; for instance, $\rho_\varepsilon = \varepsilon^{\frac{p-2}{p-1}}$ and $\sigma_\varepsilon = \varepsilon^{\frac{p-2}{2(p-1)}}$), we get

$$F_\varepsilon(\bar{w}_\varepsilon, Sv \times D_{\rho_\varepsilon}, Sv \times E_{\rho_\varepsilon}) \leq \mathcal{H}^1(Sv)\left(\gamma_p + o(1)\right) \quad \text{as } \varepsilon \to 0. \tag{5.18}$$

Step 6 : The upper bound inequality.
Now, we can use an extension lemma for the remaining pieces, which is contained in [17, Lemma 5.4].

Lemma 5.1. *Let A be a domain in $\mathbb{R}^3, A' \subset \partial A, v : A' \to [-m,m]$ a Lipschitz function (where m is given by (3.11)) and G_ε defined by (3.1).*
Then, for every $\varepsilon > 0$, there exists an extension $u : \overline{A} \to [-m,m]$ such that

$$\mathrm{Lip}(u) \leq \varepsilon^{-\frac{p-2}{p-1}} + \mathrm{Lip}(v)$$

and

$$G_\varepsilon(u,A) \leq \left((\varepsilon^{\frac{p-2}{p-1}}\mathrm{Lip}(v)+1)^p + C_m\right)\left(\mathcal{H}^2(\partial A) + o(1)\right)\omega, \quad \textit{as } \varepsilon \to 0, \tag{5.19}$$

where $C_m := \max_{t\in[-m,m]} W(t)$, $\omega := \min\{\|v-\alpha\|_{L^\infty}, \|v-\beta\|_{L^\infty}\}$.

The rest of the proof of the theorem follows one of the author [17] or Alberti, Bouchitté and Seppecher [3] with minor modifications, and we can find a Lipschitz function u_ε in the whole Ω with the required behavior. □

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E-mail address: `giampiero.palatucci@univ-cezanne.fr`

E-mail address: `sire@cmi.univ-mrs.fr`